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ON 2-BLOCKS WITH SEMIDIHEDRAL DEFECT GROUPS

BY

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ABSTRACT. Let G be one of the following groups: $L_3(q)$ and $GL(2, q)$ with $q \equiv 3 \pmod 4$, $U_3(q)$ and $GU(2, q)$ with $q \equiv 1 \pmod 4$. This paper is concerned with 2-blocks B of G having semidihedral defect groups. In particular, vertices, sources and Green correspondents of the simple modules in B are determined and used to obtain the submodule structure of the indecomposable projective modules.

Introduction. Let G be a simple group with semidihedral Sylow 2-subgroups. Alperin, Brauer and Gorenstein have shown that G must be isomorphic to one of the groups

(L) $L_3(q)$ with $q \equiv 3 \pmod 4$,

(U) $U_3(q)$ with $q \equiv 1 \pmod 4$, or

(M) M_{11};

and if $j \in G$ is an involution, then $C_G(j)$ is isomorphic to a quotient of

(C) either $GL(2, q)$ with $q \equiv 3 \pmod 4$ or $GU(2, q)$ with $q \equiv 1 \pmod 4$

by a central subgroup of odd order d (see [1]).

This paper is concerned with blocks B of these groups over fields F of characteristic 2 having semidihedral defect groups. The case M_{11} is left out; this group was already examined by J. L. Alperin (I refer to personal communication).

The first section contains required results on modular representations; some of them are more general.

In the next three sections the simple modules with vertices and sources and correspondents are determined. Summaries can be found in Tables (3.4) and (4.10). Further, some uniserial or periodic modules which are essential for the block structure are investigated.

In the fifth section, the indecomposable projectives are determined; in particular, we obtain the Cartan numbers.

The appendix contains a graphical survey of the indecomposable projectives and the Cartan matrices.

In this paper, F is assumed to be a splitting field for G and all subgroups of G. If FG is the group algebra, then J or $J(FG)$ is the Jacobson radical of FG.

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Modules are always right modules. The trivial representation is denoted by F or F_G.

Let M be an FG-module, then $l(M)$ is the number of composition factors; soc(M) is the socle, that is the largest semisimple submodule of M and head(M) is the largest semisimple factor module of M; we have head$(M) \cong M/MJ$. Further, $P(M)$ is the projective module with head$(M) \cong$ head(P); finally $\Omega(M)$ is defined by $P(M)/\Omega(M) \cong M$, and the dual M^* is the FG right module $\mathrm{Hom}_F(M, F)$.

The submodule series $M \supset MJ \supset \cdots \supset MJ^k \supset 0$ is called Loewy series. It is a submodule series of minimal length such that all factor modules are semisimple. The (unique) number $k =: j(M)$ of its factors is the Loewy length of M.

If M and M' are FG-modules, then $M \circ M'$ denotes any extension of M by M', so that there exists an exact sequence $0 \leftarrow M \leftarrow M \circ M' \leftarrow M' \leftarrow 0$. If M is indecomposable, then $vx(M)$ is the vertex and $s(M)$ a source.

For a uniserial FG-module M, we write $M = U_m(A_0, A_1, \ldots, A_{r-1})$ if $l(M) = rm$ and also $MJ^k/MJ^{k+1} = A_l$ where $k = sr + l$ and $0 < l < r$.

Let H be a subgroup of G. For any FG-module L, the induced representation $L \otimes_{FH} FG$ is denoted by L^G; and if M is an FG-module, then M_H denotes the restriction of M to FH. If M' is a summand of M_H such that $M \oplus * \cong (M')^G$, then put $f_H(M) = M'$ and $g_G(M') = M$ (if $N_G(V) < H$ where V is a vertex of M, then we have just the Green correspondence).

Concerning further terminology, we refer to Dornhoff [2], Gorenstein [6], Green [8].

1. General remarks concerning modular representations. First we collect together some of the concepts and notation related to the Green correspondence; see [7].

Let G be a finite group and char $F = p$ where p divides $|G|$. If $H < G$ and if M, M' are FG-modules, then denote by $(M, M')_H$ the vector space $\mathrm{Hom}_{FH}(M, M')$. Further $(M, M')_{H,G}$ is the subspace of $(M, M')_G$ generated by all maps of the form $\varphi = \sum \eta^g$ where $\eta \in (M, M')_H$ and where the sum is taken over a transversal of H in G. Here η^g is defined by $m \to [\eta(mg^{-1})g]$.

If ζ is a family of subgroups of G, let

$$(M, M')_{\zeta,G} := \sum_{S \in \zeta} (M, M')_{S,G}$$

and

$$(M, M')^{\zeta}_G := (M, M')_G / (M, M')_{\zeta,G}.$$

If M is an indecomposable FG-module, then there exists a p-subgroup V of G, such that M is U-projective for $U < G$ if and only if U contains a G-conjugate of V (see [2]). Then V is called a vertex of M (vx(M)).

Let $D < N_G(D) < H < G$, where D is a p-group. Define

$$\mathfrak{X} := \{D^g \cap D \mid g \notin H\},$$
$$\mathfrak{Y} := \{D^g \cap H \mid g \notin H\},$$
$$\mathfrak{A} := \{D' < D \mid \nexists\, Z \in \mathfrak{X}: D' < Z\}.$$

Let M_{FG}, N_{FH} be indecomposable FG- FH-modules, respectively, such that $\mathrm{vx}(M), \mathrm{vx}(N) \in \mathfrak{A}$. Then

$$M_H \simeq fN \oplus \mathfrak{Y}(M), \qquad N^G \simeq gN \oplus \mathfrak{X}(N),$$

where fM, gN, the Green correspondents, are indecomposable and unique up to isomorphism with the properties

$$\mathrm{vx}(M) = \mathrm{vx}(fM), \qquad gfN \simeq N,$$
$$\mathrm{vx}(N) = \mathrm{vx}(gN), \qquad fgN \simeq N,$$

$\mathfrak{Y}(M)$ [$\mathfrak{X}(N)$] is a sum of modules such that each component is Y-projective [X-projective] for a suitable $Y \in \mathfrak{Y}$ [$X \in \mathfrak{X}$].

Let M be an indecomposable $N(V)$-module where V is a vertex of M, and let $V < X < N(V)$. If $\tilde{M}$ is a nonzero indecomposable summand of M_X, then M is a summand of $\tilde{M}^{N(V)}$.

For, since M is X-projective, there is an indecomposable summand M_1 of M_X, such that M is a summand of M_1^X, and $M_1 \simeq \tilde{M} \otimes g$ for some $g \in N(V)$.

If $X = V$, then $\tilde{M}$ is called a source of M ($\tilde{M} = sM$). The following simple properties are used later without special reference:

Let M, M' be indecomposable FG-modules with vertices V, V'.

(1.1)(a) If M_X or M'_X is projective for all $X \in \mathfrak{X}$ and if M or M' is simple, then $(M, M')_{\mathfrak{X},G} = 0$.

(b) If M is simple and $M_{V'}$ is projective, then $(M, M')_G = 0$ and $(M', M)_G = 0$.

PROOF. This is a direct consequence of the facts that $(M, M')_{1,G} = 0$ if M or M' is simple (see Green [8]) and that $(M, M')_G = (M, M')_{V,G} = (M, M')_{V',G}$.

(1.2) If V and V' are not G-conjugate to subgroups $X \in \mathfrak{X}$, then $(M, M')_G^{\mathfrak{X}}$ and $(f_H M, f_H M')_H^{\mathfrak{X}}$ have the same dimension (see Green [7]).

(1.3) Let $U < G$ and ζ be a family of subgroups of U. If L is an FG-module and M and FU-module, then

(a) $\dim_F(M^G, L)_{\zeta,G} = \dim_F(M, L_U)_{\zeta,U}$,

(b) $\dim_F(L, M^G)_{\zeta,G} = \dim_F(L_U, M)_{\zeta,U}$

(see [8], [10]).

(1.4) LEMMA. *Let D be a 2-group and* char $F = 2$. *If $M = (m)$ is a submodule of FD and* $\dim_F M = \frac{1}{2}|D|$, *then* $\Omega(M^*) \simeq M$.

PROOF. As in [3, Lemma 3.2].

(1.5) LEMMA. *Let D be a 2-group, such that $D = \langle x, y\rangle$ and $|D\colon (x)| = 2$ and assume that* char $F = 2$. *Let M be an FD-module such that $M_{(x)}$ is projective. Then over D, $M = \sum_{i \in I} \oplus M_i \oplus \sum FD$, where each M_i is uniserial of dimension $|(x)|$.*

PROOF. Let $m \in M \setminus MJ$, then the $F(x)$-modules (m) and (my) are isomorphic to $F(x)$. Because $(m, my)_{(x)}$ must be projective, we have

$$(m) \cap (my)|_{(x)} = (0) \text{ or } (m). \qquad (*)$$

In the first case, $(m, my) \cong FD$. Otherwise, $\dim_F(m) = |(x)|$; and because (m) is uniserial restricted to (x), already the FD-module must be uniserial.

Finally, mFD is a direct summand, because $(*)$ holds for arbitrary elements of $M \setminus MJ$.

(1.6) LEMMA. *Let $U < G$, and let S be an irreducible FU-module, assume that S lies in the block $B(e)$.*

If $(S^G)_U \cong S \oplus X$ where $Xe = 0$, then S^G is irreducible.

PROOF. Let $M \subset \operatorname{soc}(S^G)$ and $M \neq (0)$. Then $0 \neq (M, S^G)_G \cong_F (M, S)_U$, hence $(M_U)e = S$. Now let $W := S^G/M$, then $W_U \cong (S^G)_{U/M_U}$ and therefore $(W_U)e = 0$. That is

$$0 = (S, W)_U \cong_F (S^G, W)_G$$

and so we have $W = (0)$.

(1.7) *Some periodic modules*. Let V_4 be a Klein four-group which is normal in N, then $N/C(V_4) < \mathfrak{S}_3$. Further assume that $\tilde{b}$ is a block of $C(V_4)$ with defect group V_4; let $\tilde{S}$ be the irreducible b-module.

Consider the faithful indecomposable V_4-modules of length 2. A straightforward easy calculation shows that precisely two of them are invariant under an automorphism of order 3 (they are realisable over $GF(4)$, see [3]), they are conjugate under an automorphism of order 2, and they are self-dual.

Since $\delta(\tilde{b}) = V_4$, induction and restriction yield a 1-1-correspondence between indecomposable modules of b and of FV_4. Hence $\tilde{b}$ has precisely two modules $\tilde{W} = U(\tilde{S}, \tilde{S})$, such that $\Omega\tilde{W} \cong \tilde{W}$ and such that $\tilde{W} \otimes g \cong \tilde{W}$ if $g \in N(V_4) \setminus C(V_4)$ induces an automorphism of order 3.

If $|T(\tilde{b}) : C(V_4)| < 2$, then $W := \tilde{W}^N$ is indecomposable, and $\Omega(\tilde{W}^N) \cong \tilde{W}^N$. Otherwise, let

$$C(V_4) < A < T(\tilde{b}) \quad \text{with } |A : C(V_4)| = 3,$$

then $\tilde{W}^A$ has three indecomposable summands W_1, W_2, W_3 which are complete Ω-orbits.

Since $T(\tilde{b}) = T(\tilde{S})$, we have $\tilde{S}^A = S_1 \oplus S_2 \oplus S_3$. By Brauer's first main theorem, all the S_i are in the same block.

Suppose that $\Omega W_i \cong W_i$; let $S_i = \operatorname{head}(W_i)$. Then $S_i \cong \operatorname{soc}(W_i)$, and since $l(W_i) = l(\tilde{W}) = 2$, it follows that all composition factors of $P(S_i)$ are isomorphic to S_i, that is, S_i is the only simple module in its block, a contradiction.

Hence the W_i form one Ω-orbit of length 3. Since $|T(\tilde{b}) : A|$ is a 2-power, W_i^N is indecomposable by Green's theorem (see [2]). Consequently, we have

$$W^N \cong W \oplus \Omega W \oplus \Omega^2 W$$

and (1.7*)

$$\Omega^3 W \cong W.$$

Further, $\mathrm{vx}(\Omega^i W) = V_4$.

It will be important that for $N < G$,

$$(\Omega^i W)^G \cong g(\Omega^i W) \oplus \text{projectives}$$

and (1.7**)

$$g(\Omega^i W)_N \cong (\Omega^i W) \oplus \text{projectives}.$$

More precisely, $(g\Omega^i W)_U$ is projective whenever $V_4 \not\leq_G U$. This follows from the fact that a source is projective restricted to proper subgroups of V_4.

Now let G be a group with a block B, such that $D = \delta(B)$ is semidihedral, assume that B has at least 2 simple modules.

(1.8) PROPOSITION. *Let M in B be indecomposable but not projective, $V := \mathrm{vx}(M)$, and assume that $M_{(x)}$ is projective, where $x \in D$ has order $\frac{1}{2}|D|$. Let $b = b(\varepsilon)$ be a block of $X := VC(V)$ such that $(fM)_X\varepsilon \neq 0$.*

(a) *If $|T(b) : X|$ is a 2-power, then $\Omega(M) \cong M$, and M has at least two nonisomorphic composition factors.*

(b) *If M has only one composition factor, then V is a 4-group, $|T(b) : X| = 6$ and $\Omega^3 M \cong M$.*

PROOF. Let f denote the Green correspondence between G and $N_G(V)$, then fM is an indecomposable $N_G(V)$-module with V as a vertex.

Let Y be a nonzero summand of $(fM)_X\varepsilon$. We shall prove that

$$Y \cong \Omega(Y). \qquad (1.8*)$$

This is certainly true if V is cyclic of order 2, since then there is only one nonprojective module in the block. Since M is not projective but $M_{(x)}$ is, we know that $V \not\subseteq_G (x)$. Hence we may assume that V is not cyclic.

Let sM be an indecomposable summand of Y_V, then $Y | sM^X$, and $fM | sM^{N(V)}$. Then sM must be uniserial of dimension $\frac{1}{2}|V|$, by (1.5), since V contains a subgroup (x^k) of index 2. Consequently, $sM \cong sM^*$, and therefore one obtains from (1.4) that $\Omega(sM) \cong sM$. Hence Y and ΩY are summands of sM^X in $b(\varepsilon)$.

Let $\tilde{D}$ be a defect group of $b(\varepsilon)$, then $\tilde{D} \subseteq VC(V) \cap D$ where D is a defect group of B (see [2, 57.4]). Since V is not cyclic, it follows that $\tilde{D} = V$. Thus $(sM)^X$ has exactly one summand in $b(\varepsilon)$, and therefore $Y \cong \Omega(Y)$.

If $|T(b) : X|$ is a 2-power, then $Y^{N(V)}$ must be indecomposable, by Green's theorem (see [2]), hence $\Omega(Y)^{N(V)} \cong Y^{N(V)} \cong fM \cong \Omega fM$ and therefore $M \cong \Omega M$. Because B has at least 2 simple modules, a summand of the projective cover $P(M)$ must have at least two nonisomorphic composition factors. Then $\Omega(M^*) \cong M$ forces the same for M.

If M has only one composition factor, then the fact that $|T(b) : X|$ cannot be a 2-power implies that V is either cyclic of order 2 or quaternion of order 8 or a 4-group.

Because $\Omega(M) \neq M$, the possibility $V \cong \mathbf{Z}_2$ is excluded. Then, a quaternion group of order 8 has no uniserial module of dimension 4 (see [11]), hence V is a 4-group and 3 divides $|T(b) : X|$.

Now the defect group of $b(\varepsilon)$ is V, and the block of $N(V)$ containing fM must have a bigger defect group (otherwise by the Brauer correspondence, B would have V as a defect group). Therefore $|T(b) : X|$ must be even (see, e.g., Lemma 64.7 in [2]) and is therefore 6, since this is already the order of $\mathrm{Aut}(V)$.

Let $X \subset A \subset T(b)$ such that $|A : X| = 3$. If Y^A would be indecomposable, then again by Green's theorem $fM \cong Y^N \cong \Omega(Y^N) \cong \Omega fM$ which implies that B has only one simple module, a contradiction. It follows that $Y \otimes g \cong Y$ if $g \in A \setminus X$, thus $Y \cong \tilde{W}$ if $\tilde{W}$ is as in (1.7), and from (1.7) we get that $Y^N \cong fM \oplus \Omega fM \oplus \Omega^2 fM$, where the summands are not isomorphic, and $\Omega^3 fM \cong fM$.

2. The centralizer of an involution. For the rest of the paper, G is a group with a Sylow 2-subgroup D which is semidihedral of order 2^n ($n > 4$); let

$$D := \langle x, y | x^y = x^{2^{n-2}-1}, y^2 = x^{2^{n-1}} = 1 \rangle,$$

and let j be the central involution in D. Further, we denote by V_4 a Klein four-group and by Q_r a (generalized) quaternion group of order 2^r.

In this chapter, let G be a group as in (C). Then $G = K \times O$, where $K \cong SL^{\pm}(2, q)$ with $q \equiv 3 \pmod 4$ or $K \cong SU^{\pm}(2, q)$ with $q \equiv 1 \pmod 4$, and where O is a cyclic group of odd order t with $t = (q-1)/2d$ or $(q+1)/2d$, respectively (here $d = \gcd(q-1, 3)$ or $d = \gcd(q+1, 3)$). Further, let $K_0 \lhd K$ be the subgroup of index 2, then in any case $K_0 \cong SL(2, q)$ (see [9]). Because $DC_G(D) = D \times O$, the Brauer correspondence yields that G has exactly t blocks $b_i = b_i(\varepsilon_i)$ ($i = 0, \ldots, t-1$) with D as a defect group.

We have $Z(G) = (j) \times O = O_{2',2}(G)$, so the simple modules in $b_0(G)$ and in $b_0(\bar{G})$ are the same, if $\bar{G} := G/Z(G)$. Therefore one can use the results of [3]. The irreducible modules in b_0 are $\mathrm{Irr}(b_0) = \{F_0, S_0\}$ (say) where $\dim S_0 = q - 1$, and the vertices, sources and Green correspondents are known.

Each block b_i contains a linear module, say F_i. Then

$$\mathrm{Irr}(b_i) = \{F_{i0} := F_0 \otimes F_i, S_{i0} := S_0 \otimes F_i\}$$

for all i. Moreover, $-\otimes F_i$ establishes an equivalence of the categories mod(b_0) and mod(b_i), which preserves vertices and sources and is compatible with the Green correspondence. Hence it is enough to determine the principal block.

(2.1) *The Cartan matrix*. For $X \in \operatorname{Irr}(b_0)$ let $M := P_{\overline{G}}(X)$ be the indecomposable projective $\overline{G}$-module with socle X. We may consider M as a G-module; then $\operatorname{vx}_G(M) = (j)$ and therefore $\Omega_G(M) \cong M$. Consequently, there is an exact sequence of G-modules

$$0 \leftarrow M \leftarrow P_G(X) \leftarrow M \leftarrow 0.$$

Thus we get the Cartan matrices C_0 of $b_0(G)$ from those of $b_0(\overline{G})$ which can be found in [3]:

$$C_0 = \begin{pmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+2 \end{pmatrix} \quad \text{or} \quad C_0 = \begin{pmatrix} 8 & 4 \\ 4 & 2^{n-2}+2 \end{pmatrix}$$

according to $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$. Here the first column corresponds to F_0.

If we replace G, $\overline{G}$ in (2.1) by K_0, $\overline{K}_0$, then all arguments remain true.

In $b_0(K_0)$ there are three simple modules, F_{K_0}, S_1, S_2 (say), where $S_i^K \cong S_0$ for $i = 1, 2$; and therefore $S_{0_{K_0}} \cong S_1 \oplus S_2$. Hence the Cartan matrix of $b_0(K_0)$ is

$$\begin{bmatrix} 2^{n-1} & 2^{n-2} & 2^{n-2} \\ 2^{n-2} & 2^{n-3}+2 & 2^{n-3} \\ 2^{n-2} & 2^{n-3} & 2^{n-3}+2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2^{n-3}+2 & 2^{n-3} \\ 2 & 2^{n-3} & 2^{n-3}+2 \end{bmatrix}$$

according as $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$. Here the columns correspond to F_{K_0}, S_1, S_2 in this sequence.

(2.2) LEMMA. *If X, Y are irreducible in the principal block of G or K_0, then every extension $M = X \circ Y$ has (j) in the kernel.*

PROOF. (a) If X and Y are linear, then G or K_0 (modulo Ker M) is an elementary abelian 2-group. Hence $(j) \subseteq \operatorname{Ker} M$.

(b) If X or Y are linear, then M_{G_α} or $M_{(K_0)_\alpha}$ must be semisimple (see [3]). Consequently the normal 2-subgroup (j) is contained in Ker M.

(c) In the last case, either M or $(M^G)\varepsilon_0$ is an extension $S \circ S$. If this module does not have (j) in the kernel, then it is indecomposable and the restriction to (j) must be projective. Then Proposition (1.8) yields a contradiction. Hence in each case $(j) \subseteq \operatorname{Ker} M$.

(2.3) LEMMA. *Let M be a $b_0(G)$-module, such that $M_{K_0} \subset P(F_{K_0})$. Then M is uniserial, and $M \subseteq U_r(F_0, S_0, F_0)$ where $r \leqslant 2^{n-2}$.*

PROOF. Let $P := P(F_{K_0})$, and for $r \in \mathbf{N}$ let $\operatorname{Soc}_r(= \operatorname{Soc}_r(P)) = \{m \in P \mid mJ^r = 0\}$; then Soc_r is a G-module. We prove by induction on r that Soc_r is uniserial.

From the remarks following (2.1) together with the results from [3] concerning $P(F_{\bar{K}_0})$ and (2.2) it follows that

$$\mathrm{Soc}_r/\mathrm{Soc}_{r-1} \cong \begin{cases} S_i \oplus S_2, & r \text{ even}, \\ F \oplus F, & r \text{ odd}, \end{cases}$$

$r = 2, \ldots, j(P)$.

If X is a maximal G-submodule of Soc_r, then as K_0-modules $\mathrm{Soc}_{r-1} \subseteq X \subsetneq \mathrm{Soc}_r$.

(1) Let r be even. Then $\mathrm{Soc}_r/\mathrm{Soc}_{r-1} = S_1 \oplus S_2 = (S_0)_{K_0}$ hence $X = \mathrm{Soc}_{r-1}$.

(2) Now let r be odd. Then $X/\mathrm{Soc}_{r-1} = F_0 \cong \mathrm{Soc}_r/X$. If X is not uniserial, then head $X = F_0 \oplus S_0$. This is, the K_0-module S_0/Soc_{r-2} must be semisimple, which leads to the contradiction $X \subseteq \mathrm{Soc}_{r-1}$. Finally, because $\dim(\mathrm{Ext}(S_0, F_0)) = 1$, the module Soc_r can only have a unique maximal submodule, hence Soc_r is uniserial.

Let $y \in G$ be a noncentral involution and V be the four-group $\langle j, y \rangle$; further let W be the module in the principal block of $N := N_G(V)$ which is defined in (1.7). Finally denote by N_1 the group $N_G(Y)$ and by Y the summand of $F_y^{N_1}$ in the principal block. Then W and Y^N are of the form $U_2(F_N)$; and the following holds:

(2.4) Corollary. *The modules gW and gY are uniserial of the form $U_{2^m}(F_0, S_0, F_0)$, and $gW \cong gF_N \circ gF_N$. ($q - 1 = 2^m \cdot b$, where b is odd.)*

Proof. (*) The modules gF_N, gY, gW are uniserial. The socle is isomorphic to F_0: Clearly the modules lie in the principal block, and F_0 occurs exactly once in each socle. Finally, because $(S_0)_{(y)}$ is projective, $(S_0, gY)_G = 0$ and also $(S_0, gF_N)_G = (S_0, gF_N)_{\bar{G}} = 0$; and because gW restricted to $\mathrm{vx}(S_0)$ is projective (see (1.7)), it follows that $(S_0, gW)_G = 0$.

Hence the restriction of each module to K_0 has F_{K_0} as socle, and therefore (2.3) yields (*).

Moreover, each module must be of the form $U_r(F_0, S_0, F_0)$, because it is self-dual. The fact that $\Omega gW \cong gW$ and $\Omega gY \cong gY$ yields the length for these modules.

Finally, because F_0 occurs exactly once in head(F_N^G) and head W^G and because $W^G \cong F_N^G \circ F_N^G$, we get from (*) that $gW \cong gF_N \circ gF_N$.

(2.5) Lemma. *Let k be the maximal length of a uniserial $\bar{G}$-module $U_k(S)$. Then a G-module $U_t(S)$ exists if and only if $t \leqslant 2k - 1$.*

Proof. (a) Assume that G has a module $M = U_{2k}(S)$. Then $M(1 - j) = U_t(S)$, and $M/M(1 - j) = U_{2k-t}(S)$: Both are $\bar{G}$-modules, therefore t and $2k - t$ are at most k. It follows that $t = 2k - t = k$, and therefore $M_{\langle j \rangle}$ is projective. Then Proposition (1.8) yields the contradiction that M has a linear composition factor.

(b) U_{2k-1} exists: If $q \equiv 1 \pmod 4$, then $k = 1$. So assume that $q \equiv 3 \pmod 4$. The following is true: Whenever $M \subseteq P(S)$, such that $P(S)/M$ has no linear composition factors, then $P(S)/M$ must be uniserial. Now by the preceding lemma, there is a module $Y := U(F, S, F, F, S)$. We know that $c_{FS} = 4$; since $Y \subset P(S)$ and Y^* is an epimorphic image of $P(S)$, there is a module $X \subset P(S)$ containing Y properly, such that head$(X) \cong F$. According to the Cartan matrix, $P(S)/X \cong U_{2k-1}$ is equivalent to the fact that X has three composition factors isomorphic to S. But this is true, because X is an epimorphic image of $P(F)$ which has only 4 linear composition factors.

3. **The group** $L_3(q)$. Let G be a group of type (L) or (U), and let $C := C_G(j)$; then C is a group as in §2.

Let $b_i = b_i(\varepsilon_i)$ be the blocks of C with D as a defect group ($i = 0, \dots, t-1$). The Brauer correspondence yields that G has exactly t blocks with D as a defect group. Denote them by $B_i = B_i(e_i)$, where B_i is the Brauer correspondent of b_i. Then the number of irreducibles $|\mathrm{Irr}(B_i)| = 3$ if $i = 0$ and $= 2$ otherwise; this follows from Olsson's paper **[12]**.

Now let $G = L_3(q)$, where $q \equiv 3 \pmod 4$. Then G has a 2-fold transitive permutation action on the set Ω of points in the projective plane $PG(2, q)$ of degree $|\Omega| = q^2 + q + 1$ (see e.g. **[9]**). Let G_α be the subgroup of G fixing $\alpha \in \Omega$. We may assume that $C \subset G_\alpha$; then C has a normal complement U_0 in G_α; the order $|U_0|$ is odd. Hence the C-blocks b_i may be considered as G_α-blocks with the same defect group we ientify the C-modules and G_α-modules in these blocks.

The simple B_i-modules must occur as composition factors of the modules X^G, where $X \in \mathrm{Irr}(b_i)$. Because G is 2-fold transitive on Ω, we have $G = G_\alpha \cup G_{\alpha^n}$ for $n \notin G_\alpha$. Hence it is fairly easy to compute the Mackey decomposition of $F_{i0}^G|_{G_\alpha}$. In particular, we get that $(F_0^n)^{G_\alpha}_{G_\alpha^n \cap G_\alpha}$ must have as a summand a module of $G_\alpha/U_0Z(C)$ ($\cong PGL(2, q)$), which is just the permutation representation of $PGL(2, q)$ of degree $q + 1$. Hence we get

$$\left(F_0^G|_{G_\alpha}\right) \cong F_0 \oplus U(F_0, S_0, F_0) \oplus P, \qquad (\mathrm{L}^*)$$

$$\left(F_{i0}^G\right)_{G_\alpha} \cong F_i \oplus \left(U(F_0, S_0, F_0) \otimes F_j\right) \oplus P'$$

for some $j \neq i$, if $i \neq 0$, where P, P' are projective and $P\varepsilon_0 = 0$, $P'\varepsilon_i = 0$.

(3.1) LEMMA. (a) *$F_0^G \cong F \oplus E$ where E is simple in B_0 with* $\mathrm{vx}(E) = V_4$ *and* $f_{G_\alpha}E = U(F_0, S_0, F_0)$.

(b) *For $i = 1, \dots, t$, the module $F_i := F_{i0}^G$ is simple in B_i.*

PROOF. (a) Because $|G : G_\alpha|$ is odd, F is a summand of F_0^G; say $F_0^G = F \oplus E$. Then (L*) implies that $\dim_F(F_0^G, F_0^G) = 2$, hence E is indecomposable. Again by (L*), the module E_{G_α} has a unique nonprojective summand. Because it lies in the principal block b_0, already E must be a B_0-module.

Let $(0) \neq M \subseteq E$ be simple. Because $(F, E)_G = 0$ and $(E, F)_G = 0$, neither M nor E/M can be isomorphic to F. We have

$$0 \neq (M_{G_\alpha})\varepsilon_0 = f_{G_\alpha} M \subset (E_{G_\alpha})\varepsilon_0 = U(F_0, S_0, F_0).$$

Now $M \neq F$ excludes the possibility $f_{G_\alpha} M = F_0$. If one assumes that $f_{G_\alpha} M = U(S_0, F_0)$, then one gets $f_{G_\alpha}(E/M) = F_0$ and the contradiction that $E/M = F$.

Therefore $f_{G_\alpha} M = (E_{G_\alpha})\varepsilon_0 = f_{G_\alpha} E$, that is $M = E$. As we have just remarked, $f_{G_\alpha}(E)$ is a permutation module. Thus the vertex is a Sylow 2-subgroup of the corresponding stabilizer (see [13]); this is a Klein 4-group.

(b) follows directly from (L^*) and Lemma (1.6).

(3.2) LEMMA. *The modules $S := S_0^G$ and $S_i := S_{i0}^G$ are simple.*

PROOF. By (1.6) it is enough to show that

$$(S_{i0}^G)_{G_\alpha}\varepsilon_i \cong S_{i0} \quad \text{for } i = 0, \ldots, t.$$

Clearly S_{i0} is a summand of $(S_{i0}^G)_{G_\alpha}$. Further, $(F_i, S_{i0})^G = 0$ and $(S_{i0}^G, F_i) = 0$, and finally

$$\dim(S_i, S_{i0}^G) = 1 + \dim(S_{i0}, S_{i0}^n)_{G_\alpha^n \cap G_\alpha} \quad \text{for } n \notin G_\alpha.$$

So we have to show that $(S_{i0}, S_{i0}^n)_{G_{\alpha\beta}} = 0$ if $n \notin G_\alpha$ and $\alpha^n = \beta$.

(a) $(S_{i0})_{G_{\alpha\beta}}$ is simple: Modulo $\mathrm{Ker}(S_0)$, the group $G_{\alpha\beta}$ is isomorphic to a subgroup of $PGL(2, q)$ of index $q + 1$ fixing a point in $PG(1, q)$. This is a Frobenius group of order $(q-1) \cdot q$. Now S_0 is a faithful module of this group of dimension $q - 1$, hence is simple. It follows that (a) is true.

(b) $S_{i0} \ncong S_{i0}^n$. One can find an element $k \in \mathrm{Ker}(S_{i0})$ but $k \notin \mathrm{Ker}(S_{i0}^n)$. Because S_{i0} and S_{i0}^n are nonisomorphic simple $G_{\alpha\beta}$-modules, $(S_{i0}, S_{i0}^n)_{G_{\alpha\beta}} = 0$.

(3.3) COROLLARY. (a) *A module $U_t(S_i)$ exists if and only if $t < 2^{n-2} - 1$.*

(b) *If $i \neq 0$, then $g_G(U_2(F_{i0}, S_{i0}, F_{i0})) = U_2(F_i, S_i, F_i)$.*

PROOF. By (2.5), (3.1) and (3.2).

(3.4) *Vertices and correspondents of simple modules.*

simple module	f_{G_α}	vertex	source	Green correspondent
F	F_0	D	F_D	F_D
F_i	F_{i0}	D	F_D	F_i
S	S_0	Q_{n-1}	$U_{2^{n-3}-1}(F_{Q_{n-1}})$	$U_{2^{n-3}-1}(F_{Q_{n-1}})^D$
S_i	S_{i0}	Q_{n-1}	$U_{2^{n-3}-1}(F_{Q_{n-1}})$	$U_{2^{n-3}-1}(F_{Q_{n-1}})^D \otimes F_i$
E	$U(F_0, S_0, F_0)$	V_4	F_{V_4}	$U_2(F_{N_G(V_4)})$

($i = 1, 2, \ldots, (q-1)/2d - 1$). The G_α-correspondents in this table are just determined; all remaining entries are taken from [3].

Let V be a 4-group, $N := N_G(V)$ and $W \in b_0(N)$ be the module defined in (1.8).

(3.5) LEMMA. $gW = U(E, E)$, $g\Omega W = U(F, S, F, E)$, *and* $g\Omega^2 W = U(E, F, S, F)$.

PROOF. Let $N_1 := N_C(V)$ and let $\tilde{W}$ be a summand of W_{N_1}, then $\tilde{W} = U_2(F_{N_1})$. We have $|N : N_1| = 3$, and by (1.7),

(i) $\tilde{W}^N = W \oplus \Omega W \oplus \Omega^2(W)$.

As we have shown in Lemma (2.4), there is an exact sequence of $b_0(C)$-modules.

(ii) $0 \leftarrow g_C F_{N_1} \leftarrow g_C \tilde{W} \leftarrow g_C F_{N_1} \leftarrow 0$ and $g_C(\tilde{W}) = U_2(F_0, S_0, F_0)$, $g_C(F_{N_1}) = U(F_0, S_0, F_0)$.

We may consider (ii) as a sequence of G_α-modules. Then we know the composition factors of $X := U(F_0, S_0, F_0)^G$, and we can determine $(X, Y)_G$ and $(Y, X)_G$ for all simple G-modules Y (using (3.1), (3.2)). This yields straightforwardly that

(iii) $X = E \oplus E \oplus U(F, S, F)$.

Now (i) implies that $(g_C \tilde{W})^G = gW \oplus g\Omega W \oplus g\Omega^2 W$: moreover gW is the unique self-dual summand. By (ii) and (iii), the only possibility is that stated.

(3.6) *The Cartan matrices.* If P is a projective G_α-module, then P^G is projective. Hence it follows from (3.1)(b) and (3.2) that the G-blocks B_i ($i \neq 0$) and the block $b_0(G_\alpha)$ have the same Cartan matrix. Now let $i = 0$. Because $(E_{G_\alpha})\varepsilon_0 = U(F_0, S_0, F_0)$ and $(S_{G_\alpha})\varepsilon_0 = S_0$ (see (3.1)), it follows that

$$P(F_0)^G \cong P(F) \oplus P(E) \oplus P(E),$$

and

$$P(S_0)^G \cong P(S) \oplus P(E).$$

We know the composition factors of $P(F_0)^G$ and of $P(S_0)^G$ (by (2.1), (3.1), (3.2)) and those of $P(E)$ (by (3.5)). Hence we obtain the Cartan matrix of B_0, as stated in the Appendix.

4. The group $U_3(q)$. Let $G = U_3(q)$ where $q \equiv 1 \pmod 4$. Then G has a 2-fold transitive permutation action on the set Ω of isotropic points in $PG(2, q^2)$, of degree $|\Omega| = q^3 + 1$. Let G_α be the subgroup fixing a point $\alpha \in \Omega$. Then $G_\alpha = U \cdot H = N_G(U)$, where U is a Sylow p-subgroup of G (if $p^l = q$) of order q^3 and H is cyclic of order $(q^2 - 1)/d$.

We may assume that $H = O \times \tilde{H}$ if $O = O_2$, (C) is as before, $|O| = t$.

Let F_{i1} ($i = 0, \ldots, t - 1$) be the irreducible representations of O, considered as G_α-modules; $F_{i1}|_{B \cap C} = F_{i0}|_{B \cap C}$. Further, G_α has exactly d nonlinear irreducible modules L_i having the centre $Z(U)$ in the kernel, and $\dim L_i = (q^2 - 1)/d$.

We shall find the simple modules in the blocks B_i as composition factors of F_{i1}^G. The Mackey decomposition yields

$$(F_{i1}^G)_{G_\alpha} \cong F_{i1} \oplus F_{i1} \oplus \sum_{k=1}^{d} \oplus L_k \oplus R \qquad (\mathrm{U}^*)$$

where $Z(U)$ is not in the kernel of any composition factor of R.

Secondly it will be important to consider $F_{i1}^G|_C$. The group C has two orbits on Ω, where $|\alpha^C| = q + 1$; further a noncentral involution $y \in C$ fixes a point $\beta \notin \alpha^C$. Now F_{01}^G is just the permutation module $F\Omega$, hence

$$F_{01}^G|_C \cong F_{C_\alpha}^C \oplus F_{C_\beta}^C, \qquad (\mathrm{U}^{**})$$

where $F_{C_\alpha}^C \cong U(F_0, S_0, F_0)$ and $F_{C_\beta}^C \cong \sum_{i=0}^{t-1} gY \otimes F_i \oplus P$; here $gY = U_{2^n-2}(F_0, S_0, F_0)$ is the module determined in (2.4), and $P\varepsilon_i = 0$ for $i = 0, 1, \ldots, t$.

Similarly, one obtains for arbitrary i

$$(F_{i1}^G)_C\varepsilon_i \cong U(F_{i0}, S_{i0}, F_{i0}) \oplus (gY \oplus F_i). \qquad (\mathrm{U}^*_{**})$$

(4.1) LEMMA. *The modules F_{i1}^G are indecomposable with simple socles and heads* ($i = 0, \ldots, t - 1$).

PROOF. By (U*), $\dim(F_{i1}^G, F_{i1}^G) = 2$. Then we know that $F_{i1}^G = M \oplus M'$, where M is indecomposable with $\mathrm{vx}(M) = \langle x \rangle \in \mathrm{Syl}_2(G_\alpha)$. Because the block B_i containing M has a noncyclic defect group, M cannot be simple (see [4]). Finally, applying (1.1)–(1.3) yields

$$l(\mathrm{soc}(F_{i1}^G) \oplus \mathrm{head}(F_{i1}^G)) = 2.$$

This implies the statement.

(4.2) LEMMA. $\mathrm{Soc}(F_{01}^G) \cong F \cong \mathrm{head}(F_{01}^G)$; *and* $(F_{01}^G)J/F \cong S \oplus E$, *where* S, E *are simple with* $f_C S = S_0$ *and* $f_C E = U_{2^n-2}(F_0, S_0, F_0)$.

PROOF. By the preceding, $\mathrm{soc}(F_{01}^G) \cong F \cong \mathrm{head}(F_{01}^G)$. It remains to determine $X := (F_{01}^G)J/F = F\Omega J/F$. From (U**) and from $\mathrm{Ext}(S_0, F_0) = F$ (see (2.2)) it follows that $(X_C)\varepsilon_0 \cong S_0 \oplus M$ where M is uniserial, $M = U_{2^n-2}(F_0, S_0, F_0)$.

Suppose X is indecomposable.

Because $\mathrm{vx}(S_0) = Q_3$ (see [3]), it follows that $f_C X \cong S_0$ and M is $\mathcal{Y}$-projective, where

$$\mathcal{Y} = \{Q_3^g \cap C | g \notin C\} = \{(1), (y) | y^2 = 1, y \notin (j)\}.$$

Hence $\mathrm{vx}(M) = (y)$, and $M \cong gY \otimes F_0$. Consequently $X_C \cong S_0 \oplus F_{C_\beta}^C$, hence $X_{(y)}$ has exactly $(q + 1)$ summands isomorphic to $F_{(y)}$.

But this yields a contradiction: Firstly we have the vector space decomposition $F\Omega = \beta F \oplus F\Omega J$. Secondly, from $F\Omega J = \sum_{\omega_i \in \Omega \setminus \{\beta\}} F(\omega_i - \beta)$ and from $q + 1 = |\mathrm{fix}_\Omega(y)| > 1$ we see that $\Sigma_{\omega\in\Omega}\, \omega \notin (F\Omega J)(1 - y)$. Therefore $X_{(y)}$ has only $q - 1$ summands $F_{(y)}$.

Each summand of X provides at least one summand of $(X_C)\varepsilon_0$, hence we have shown

(I) $X \cong S \oplus E$, where S and E are nonzero indecomposable B_0-modules.

Assume that $L_1 \subseteq E_{G_\alpha}$; then $L := \Sigma_{i=1}^d L_i \subseteq E_{G_\alpha}$, because E must be invariant under algebraic conjugation.

(II) E is simple. Let N be the composition factor of E, such that $L \subseteq N_{G_\alpha}$. Then $U_{2^{n-1}}(F_H)$ is a summand of N_H, hence N_C must have 2^{n-1} composition factors isomorphic to F_0. But the unique submodule of X_C minimal with this property is M. It follows that $f_C N = (N_C)\varepsilon_0 \cong M$ and then $M \cong (E_C)\varepsilon_0 = f_C E$, therefore $N \cong E$.

(III) S is simple, because $(S_C)\varepsilon_0 = S_0$ and S_0 is simple.

(4.3) COROLLARY. *Let* $P := P(F)$; *then* $j(P) = 2^n + 1$ *and*

$$PJ^k/PK^{k+1} \cong \begin{cases} S \oplus E & \text{if } k \text{ is odd,} \\ F \oplus F & \text{if } k \text{ is even,} \end{cases}$$

$(k = 1,2, \ldots, 2^n - 1)$.

PROOF. The G_α-module $U_{2^{n-1}}(F_{G_\alpha})$ is projective, hence the preceding lemma implies that $U_{2^{n-1}}(F_{G_\alpha})^G \cong P$. Because G is a nonabelian simple group, $\mathrm{Ext}(F, F) = 0$. By induction it follows that $P_r := U_r(F_{G_\alpha})^G$ has Loewy length $2r + 1$ and that the Loewy factors are as stated.

(4.4) COROLLARY. (a) *The vertex of E is a four-group*; *and a source is faithful of dimension* 2.

(b) $\Omega^3 E \cong E \cong E^*$.

PROOF. In (4.2) we have seen that $E_{(x)}$ is projective. Apply (1.8).

(4.5) LEMMA. *If* $i \neq 0$, *then* $F_{i1}^G = U(F_i, S_i, F_i)$ *where* $f_C F_i = F_{i0}$ *and* $f_C S_i = S_{i0}$ *and* $\{F_i, S_i\} = \mathrm{Irr}(B_i)$.

PROOF. Let $X := \mathrm{soc}(F_{i1}^G)$ and $Y := \mathrm{head}(F_{i1}^G)$. Further, let S_i be the composition factor of F_{i0}^G, such that $L \subseteq (S_i)_{G_\alpha}$; of course $S_i \neq X, Y$.

Then F_{i0} occurs in $(X_C)\varepsilon_i$ and in $(Y_C)\varepsilon_i$ exactly once, and $(S_i)_C\varepsilon_i$ must contain $U_{2^n}(F_{i0}, S_{i0}, F_{i0})$, see (U^*_{**}).

Assume that $(S_i)_C\varepsilon_i = U_{2^n}(F_{i0}, S_{i0}, F_{i0})$. Then $(S_i)_{(x)}$ is projective, hence Proposition (1.8)(b) implies that $V := \mathrm{vx}(S_i)$ must be a four-group and that $|T(b) : C(V)| = 6$ if b is a $C(V)$-block containing a summand of $(fS_i)_{C(V)}$. But then b is the principal block and therefore $i = 0$, contradiction. Hence

$$(S_i)_C\varepsilon_i \cong S_{i0} \oplus U_{2^n}(F_{i0}, S_{i0}, F_{i0}) \quad \text{and}$$
$$(X)_C\varepsilon_i \cong F_{i0} \cong (Y_C)\varepsilon_i. \qquad (4.5^*)$$

So $X \cong Y =: F_i$, and F_{i1}^G has only these composition factors.

REMARK. The modules F_{i1}^G are liftable; let π_i be the corresponding character. Each block B_i contains irreducible characters χ_i, ρ_i, where $\chi_i(1) = q(q-1) + 1$ such that $\chi_i + \rho_i \equiv \pi_i \pmod 2$ (see e.g. [5]). Hence $\dim S = q(q-1)$, $\dim E = (q^2+1)(q-1)$; then, for $i \neq 0$, $\dim F_i = q(q-1)+1$ and $\dim S_i = (q-1)(q^2-q+1)$.

Now we induce simple C-modules.

(4.6) LEMMA. *Let* $i \in \{0, \ldots, t-1\}$.

(a) $(F_{i0})^G e_i \cong F_i \oplus M$, *where* $\mathrm{soc}(M) \cong \mathrm{head}(M) \cong S_i$ *and* $\mathrm{vx}(M)$ *is a four-group.*

(b) $(S_{i0}^G)e_i \cong S_i$.

PROOF. Computing homomorphisms yields (b) and also (a) apart from the last statement.

We know that $(F_i)_C \varepsilon_i = F_{i0}$. Now because $N_C(V_4) \not\subseteq C$, the Mackey decompositions imply that $(F_{i0}^G|_C)\varepsilon_i$ has a summand with V_4 as vertex. Consequently $(F_{i0}^G)e_i$ must have a component with such a vertex, this can only be M.

Let $M := (g_C W) \otimes F_i$, where W is the module defined in (1.8), then $M = U_{2^{n-2}}(F_{i0}, S_{i0}, F_{i0})$, by (2.4).

For $k = 1,2, \ldots, 2^{n-2}$ denote by M_k the submodule $U_k(F_{i0}, S_{i0}, F_{i0})$ of M.

(4.7) LEMMA. *For* $k = 1,2, \ldots, 2^{n-2}$ *and* $i > 1$, *the module* $g_G(M_k)$ *is uniserial of the form* $U_k(F_i, S_i, F_i)$.

PROOF. First we show:

$$(M_1^G)e_i = g_G M_1 \oplus P(S_i) \quad \text{for arbitrary } i. \tag{4.7*}$$

The vertex of M_1 is not conjugated to a subgroup of $\mathfrak{X}$ ($\dim M_1 = q + 1 \not\equiv 0 \pmod 4$). So there is a unique summand $g_G M_1$ which must lie in B_i.

From (4.6) one concludes that $\mathrm{head}(M_1^G e_i) \cong F_i \oplus S_i \cong \mathrm{Soc}(M_1^G e_i)$. Finally, by (4.6) we have that $(M_1, S_i)_C = (M_1, S_i)_{\mathfrak{X},C}$. From the properties of fS_i (see (4.5) and [3]), it follows that any $\mathfrak{X}$-projective summand of $(S_i)_C$ must be (y)-projective for a noncentral involution $y \in D$.

By (1.3), also $(M_1^G, S_i)_G = (M_1^G, S_i)_{(y),G} = (M_1^G, S_i)_{1,G}$ since $(S_i)_{(y)}$ is projective. That is, a homomorphism $0 \neq \varphi \in (M_1^G, S_i)_G$ must factorise over $P(S_i)$; this yields (4.7*).

The vertex of M_k is either not conjugated to a subgroup of $\mathfrak{X}$ or is a four-group V. In the latter case, if b_i is a block of $C(V)$ containing a summand of $(fM_k)_{C(V)}$, then $T(b_i) <_G C$ provided $i \neq 0$.

Hence if $i \neq 0$, then $g_G(M_k)$ is always unique. We conclude that

(a) $(M_k^G)e_i = gM_k \oplus k \cdot P(S_i)$ if $i \neq 0$.

Now we can prove the statement. Knowing the socle, head and composition factors we get

(b) $gM_1 = U(F_i, S_i, F_i)$.

So let $1 < k \leqslant 2^{n-2}$. If one induces the exact sequence of b_i-modules

$$0 \leftarrow M_{k-1} \leftarrow M_k \leftarrow M_1 \leftarrow 0$$

one obtains in B_i an exact sequence

$$0 \leftarrow gM_{k-1} \oplus (k-1)P(S_i) \leftarrow gM_k \oplus k \cdot P(S_i) \leftarrow gM_1 \oplus P(S_i) \leftarrow 0.$$

Hence $gM_k \cong gM_{k-1} \circ gM_1$.

By the induction hypothesis, $gM_{k-1} = U_{k-1}(F_i, S_i, F_i)$. Assume that gM_k is not uniserial. Then it contains a submodule $U_2(F_i)$, and we obtain the contradiction $2 \leqslant \dim(U_2(F_i), M_k^G)_G = \dim(U_2(F_i), M_k)_C = 1$.

(4.8) COROLLARY. *For arbitrary i, the Cartan number $c_{S_iS_i}$ is even.*

PROOF. If $(F_{i0}^G)e_i = F_i \oplus M$ as in (4.6), then (4.7*) implies that $M \circ M$ and $P(S_i)$ have the same composition factors.

(4.9) LEMMA. $\mathrm{Ext}(S_i, S_i) = 0$ *for arbitrary i.*

PROOF. Let $M = S_i \circ S_i$, then $(M_C)\varepsilon_i \cong (S_{i0} \oplus X) \circ (S_{i0} \oplus X)$ where $X = U_{2^n-2}(F_{i0}, S_{i0}, F_{i0})$ (see (4.5*)). From (2.3) it follows that there is no indecomposable module $S_{i0} \circ X$ or $X \circ S_{i0}$. The consequence is that $(M_C)\varepsilon_i \cong S_{i0} \oplus S_{i0} \oplus R$. Now $\mathrm{vx}(S_{i0})$ is quaternion; hence M cannot be indecomposable.

(4.10) *Vertices and correspondents of simple modules.*

simple module	f_C	vertex	source	Green correspondent
F	F_0	D	F_D	F_D
F_i	F_{i0}	D	F_D	F_i
S	S_0	Q_3	$U_2(F_{Q_3})$	W
S_i	S_{i0}	Q_3	$U_2(F_{Q_3})$	$W \otimes F_i$
E	$U_{2^n-2}(F_0, S_0, F_0)$	V_4	$U_2(F_{V_4})$	W

$(i = 1,2, \ldots, (q+1)/2d - 1)$.

Here, W is the module defined in (1.7) which is considered as an $N_G(Q_3)$-module in the rows 3 and 4.

The C-correspondents in this table have just been determined; all other entries are taken from [3].

5. Indecomposable projective modules. In this chapter, W is always the $N_G(V_4)$-module with $\Omega^3 W \cong W$ defined in (1.7), such that gW lies in the block which is just considered.

First we determine for blocks with three simple modules the projective $P := P(E)$.

(5.1) THEOREM. (a) *Let G be of type* (L). *Then $j(P) = 5$, and $PJ/E \cong E \oplus U(F, S, F)$.*

(b) *Let G be of type* (U). *Then P is uniserial of length $2^n + 1$, and $PJ \cong U_{2^n-2}(F, S, F, E)$.*

PROOF. We have shown in (3.5), (4.4) that $gW = U(E, E)$ if $G = L_3(q)$, and $gW = E$ if $G = U_3(q)$. Moreover, (a) follows from (3.5).

Now let G be of type (U). Knowing $f(\Omega E)$ and $\mathrm{vx}(S)$, the application of (1.1)–(1.3) yields $\mathrm{head}(\Omega E) \cong F$. Hence one has a short exact sequence

$$0 \leftarrow \Omega E \leftarrow P(F) \leftarrow (\Omega E)^* \leftarrow 0 \qquad (\mathcal{F})$$

and therefore, because all simple B_0-modules are self-dual, c.f. $P(E) = E + \frac{1}{2}$ c.f. $(P(F))$. In particular, the length of P is $2^n + 1$. However, by (4.3), the Loewy length of P must be at least $2^n + 1$, and therefore P is uniserial. Consequently $(\mathcal{F})$ yields

(i) $PJ^k/PJ^{k+1} \cong F$ if k is odd, and $\cong S$ or E if k is even.

Because $c_{SE} \neq 0$,

(ii) $PJ^2/PJ^3 \cong S$, and therefore

(iii) P cannot have a factor isomorphic to $U(E, F, E)$.

Now E and S occur in PJ with the same multiplicity, and $P \cong P^*$, therefore (iii) implies

(iv) P cannot have a factor isomorphic to $U(S, F, S)$.

Hence the only possibility is that stated.

In the following, we write F_i for F_{i0} and S_i instead of S_{i0} if G is of type (C), and sometimes F_0 for F and S_0 for S. Remember that $q - 1 = 2^m \cdot b$ where b is odd. Now let $P := P(F_i)$ for arbitrary i. Then we have:

(5.2) THEOREM. (a) *Let G be of type* (C) *or let G be of type* (L) *or* (U) *and $i \neq 0$. Then the Loewy length of P is $j(P) = 3 \cdot 2^m + 1$. A Loewy series of P is given by*

$$PJ^k/PJ^{k+1} \cong \begin{cases} S_i \oplus F_i & \text{if } k \equiv 1 \text{ or } 2 \pmod 3, \\ F_i \oplus F_i & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

$(k = 1,2, \dots, j(P) - 2)$.

(b) *Let G be of type* (L) *or* (U) *and let $i = 0$. Then the Loewy length $j(P)$ is 5 if $G = L_3(q)$ or $2^n + 1$ if $G = U_3(q)$. A Loewy series of P is given by*

$$PJ^k/PJ^{k+1} \cong \begin{cases} S \oplus E, & k \text{ odd}, \\ F \oplus F, & k \text{ even} \end{cases}$$

$(k = 1,2, \dots, j(P) - 2)$.

PROOF. There is an exact sequence

$$0 \leftarrow g\Omega W \leftarrow P \leftarrow g\Omega^2 W \leftarrow 0.$$

The statements follow straightforwardly from (2.2) if G is of type (C), and from (2.2) and (3.3)(b) or (4.7) if G is of type (L) or (U) and $i \neq 0$. Finally, (b) follows from (3.5) and (4.3). If P is indecomposable projective with socle X, then denote by $H(X)$ ("heart") the module PJ/X.

(5.3) LEMMA. (i) *The module* $H(F_i)$ *is indecomposable.*

(ii) $H(F_i)$ *contains a maximal submodule* $H_1 \oplus H_2$, *where* H_i *is uniserial and* $l(H_1) + 1 = l(H_2)$.

In case (5.2)(a), $H_1 = U(F_i, F_i, S_i, \ldots, F_i)$ *and* $H_2 = U(F_i, S_i, F_i, \ldots, S_i)$. *If* $i = 0$ *and* G *is of type* (L), *then* $H_1 = U(F, E)$ *and* $H_2 = U(F, E, S)$. *If* $i = 0$ *and* G *is of type* (U), *then* $H_1 = U(F, E, F, S, \ldots, F, E)$ *and* $H_2 = U(E, F, S, F, \ldots, E, F, S)$.

(iii) *If* $H_1 \subset M \subset H(F_i)$, *but* $M \not\subseteq H(F_i)J$, *and* head$(M) \cong S_i$, *then* $M \cap H_2 \cong S_i$.

PROOF. (ii) Using the modules $g\Omega^2W$ and $((g\Omega^2W)J^2)^*$ one can construct an extension $(H_1 \oplus H_2) \circ F_i \subset P(F_i)$ as stated.

(iii) If M would be uniserial of length $1 + l(H_1)$, then it follows that $\Omega M = U_2(S_i)$, contradiction. On the other side, $0 \neq \Omega M \subset P(S_i)$, so the only possibility is that $\Omega M \cong S_i$ and M is as stated.

(i) follows from (iii).

Now we determine $P := P(S_i)$. Assume first that $q \equiv 1 \pmod 4$.

(5.4) THEOREM. (a) *Let* G *be of type* (C) *or of type* (U) *where* $i \neq 0$. *Then* $j(P) = 3 \cdot 2^{n-2} + 1$, *and* P *has the Loewy series*

$$PJ^k/PJ^{k+1} \cong \begin{cases} F_i & \text{if } k \neq 2 \text{ and } k \not\equiv 0 \pmod 3, \\ F_i \oplus S_i & \text{if } k = 2, \\ S_i & \text{if } k \equiv 0 \pmod 3 \end{cases}$$

$(k = 1,2, \ldots, j(P) - 2)$.

(b) *Let* G *be of type* (U) *and* $i = 0$. *Then* $j(P) = 2^n + 1$, *and* P *has the Loewy series*

$$PJ^k/PJ^{k+1} \cong \begin{cases} F & \text{if } k \text{ is odd}, \\ E \oplus S & \text{if } k = 2, \\ E & \text{if } k = 4r, \\ S & \text{if } k = 4r + 2 \end{cases}$$

$(k = 1, 2, \ldots, j(P) - 2)$.

PROOF. In this proof, we omit the index "i". Let M be the uniserial G-module $(g\Omega W)J$, then head$(M) = S$, by (2.2), (4.7) and (5.1). Hence there is an exact sequence

$$0 \leftarrow M \leftarrow P \leftarrow \Omega M \leftarrow 0. \qquad (5.4^*)$$

The composition series of M is known. We show that

$$\Omega M = U(S, F, S).$$

From the existence of a module $U(F, F, S)$ (or $U(E, F, S)$ in (b)) in the block one obtains the statement of the theorem.

Because $\mathrm{soc}(\Omega M) \cong S$ and $\mathrm{Ext}(S, S) = 0$ [see (2.2), (4.9)] and because F occurs exactly once in ΩM – c_{FS} is known –, it follows that $\Omega M \subseteq U(S, F, S)$. Finally the fact that c_{SS} is even [see (2.1), and (4.8)] yields equality.

(5.5) LEMMA. (i) *The modules $H(S_i)$, $H(S)$ are indecomposable.*

(ii) *There are submodules $H_1 = U(F_i, S_i, F_i, \ldots, F_i)$ of lenth $3 \cdot 2^{n-2} - 1$ and $H_2 = U(S_i, F_i)$ with $H_1 \cap H_2 = F_i$ in case* (5.4)(a) *and $H_1 = U(E, F, S, F, \ldots, E, F)$ of length 2^{n-1} and $H_2 = U(S, F)$ with $H_1 \cap H_2 = F$ in case* (5.4)(b).

(iii) *The submodules $H_1 \circ S_i$, $H_2 \circ S_i$ of $P(S_i)$ are maximal uniserial.*

PROOF. (ii) The modules $g\Omega^2 W/F$ and ΩM from (5.4*) contained in $P(S_i)$ or $P(S)$ enable us to construct extensions $H_1 + H_2/S_i$ or $H_1 + H_2/S$, as stated.

(i) follows from (ii). Knowing the Cartan numbers and all extensions of simple modules, one obtains (iii) straightforwardly. Finally, let $q \equiv 3 \pmod 4$.

(5.6) THEOREM. (a) *Let G be of type* (C) *or of type* (L) *where $i \neq 0$. Then $j(P) = 2^{n-2} + 1$; and a Loewy series of P is*

$$PJ^k/PJ^{k+1} \cong \begin{cases} F_i \oplus S_i, & k = 1, 2, 4, 5, \\ S_i \oplus S_i, & k = 3, \\ S_i, & \textit{otherwise} \end{cases}$$

$(k = 1, \ldots, j(P) - 2)$.

(b) *Let G be of type* (L) *and $i = 0$. Then $j(P) = 2^{n-2} + 1$, and a Loewy series of P is*

$$PJ^k/PJ^{k+1} \cong \begin{cases} F \oplus S, & k = 1, 3, \\ E \oplus S, & k = 2, \\ S, & \textit{otherwise} \end{cases}$$

$(k = 1, 2, \ldots, j(P) - 2)$.

PROOF. We omit the index "i" in this proof. Let $X := g(\Omega W)/F$. Then there is an indecomposable G-module $M := U_{2^{n-2}-1}(S) \oplus X/S$, such that $\mathrm{soc}(M) \cong S$, so $M \subset P$. Now P/M has composition factors F, S, S [see (2.1), (3.6)]. It follows that $(P/M)J = F \oplus S$, and $P/M \cong P/PJ^2$. So we obtain the results.

In this situation, we have

(5.7) LEMMA. (i) *The module* $H(S_i)$ *is indecomposable.*
(ii) $H(S_i)J \cong H_1 \oplus H_2$, *where the* H_i *are uniserial.*
In case (5.6)(a), $H_1 = U(F_i, S_i, F_i, F_i)$ *and* $H_2 = U_{2^{n-2}-2}(S_i)$.
In case (5.6)(b), $H_1 = U(E, F)$ *and* $H_2 = U_{2^{n-2}-2}(S)$.
(iii) *If* $H_1 \subsetneqq M \subset H(S_i)$ *and* head$(M) \cong F_i$, *then* $M \cap H_2 = S_i$.

PROOF. (ii) Using the modules from (2.5), (3.3), (3.5) one can construct an extension $(H_1 \oplus H_2) \circ S_i \subset P(S_i)$ or $(H_1 \oplus H_2) \circ S \subset P(S)$ as stated.
(iii) If M would be uniserial, then one obtains a contradiction to (2.5), (3.3) respectively.
(i) follows from (iii).

Appendix. In order to describe an indecomposable projective FG-module, we associate to P a matrix which has in the kth row the composition factors of PJ^k/PK^{k+1}, with multiplicities.

If there is an arrow down from a composition factor X in the kth row, this means that a submodule M of PJ^k, $M \not\subseteq PJ^{k-1}$ with $X \subseteq$ head M must contain the composition factor at the end of the arrow.

TABLE 1

Type of G	Blocks	$P(F)$	$P(S)$	$P(E)$	Cartan matrix
(C) $q \equiv 1 \pmod 4$ (U) $q \equiv 1 \pmod 4$	b_i $\left(i = 0, \dots, \frac{q+1}{2d} - 1\right)$ $B_{i'}$ $\left(i = 1, \dots, \frac{q+1}{2d} - 1\right)$	F S F F S F F S F F S F F $\vdots$ $\vdots$ S F F S F	S F F S F F S S $\vdots$ F F S	---	F S $\begin{pmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+2 \end{pmatrix}$
(C) $q \equiv 3 \pmod 4$ (L) $q \equiv 3 \pmod 4$	b_i $\left(i = 0, \dots, \frac{q-1}{2d} - 1\right)$ $B_{i'}$ $\left(i = 1, \dots, \frac{q-1}{2d} - 1\right)$	F S F F S F F S F F S F	S F S S F $\vdots$ S $\vdots$ F F S S	---	F S $\begin{pmatrix} 8 & 4 \\ 4 & 2^{n-2}+2 \end{pmatrix}$

TABLE II

Type of G	Blocks	$P(F)$	$P(S)$	$P(E)$	Cartan matrix
(U) $q \equiv 1 \pmod 4$	B_0	F $S \quad E$ $F \quad F$ $E \quad S$ $F \quad F$ $\cdot \quad \cdot$ $\cdot \quad \cdot$ $\cdot \quad \cdot$ $\cdot \quad \cdot$ $\cdot \quad \cdot$ $\cdot \quad \cdot$ $F \quad F$ $E \quad S$ F	S F E F S $\cdot$ $\cdot$ $\cdot \quad S$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ F E F S	E F S F $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ $\cdot$ F S F E	$\begin{matrix} F & S & E \end{matrix}$ $\begin{pmatrix} 2^n & 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+2 & 2^{n-2} \\ 2^{n-1} & 2^{n-2} & 2^{n-2}+1 \end{pmatrix}$
(L) $q \equiv 3 \pmod 4$	B_0	F $S \quad E$ $F \quad F$ $E \quad S$ F	S S $F \quad S$ $\cdot \quad \cdot$ $E \quad \cdot$ $\cdot \quad \cdot$ $F \quad \cdot$ $\cdot$ S S	E F $S \quad E$ F E	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2^{n-2}+1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

If there is not such an arrow, then there exists a uniserial module $M \subset PJ^k$, $M \not\subseteq PJ^{k+1}$ with head$(M) = X$, where the composition series of M is given by the composition factors underneath, including the socle.

Here, the irreducible modules in a block B_i are denoted by F, S or F, S, E respectively. Finally, $d =$ g.c.d.$(3, q-1)$ or g.c.d.$(3, q+1)$ according as $G < L_3(q)$ or $G < U_3(q)$, and 2^n is the order of a Sylow 2-subgroup of G.

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